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On the Theory of Mixed-Path Ground-Wave Propagation on a Spherical Earth[1]

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The problem formulated concerns the mutual impedance between two vertical dipole antennas A and B located near the surface of a spherical smooth earth. The path between A and B is made-up of a number of homogeneous segments where the surface impedance is constant. Various formulas are developed, for two- and three-section paths, which are suitable for computation. Certain limiting cases are discussed and where possible a physical interpretation of the results is given. Comparisons with previous work are made.

1. Introduction

In the computation of the ground-wave field of a radio transmitter it is often assumed that the earth is a homogeneous and smooth conducting sphere. While this is a satisfactory approach for many applications, such as estimating the coverage of broadcast transmissions, there is often a need for more precise determinations. For example, the phase characteristics of the ground wave are significantly influenced by inhomogeneities in conductivity that occur at coast lines. This is an important consideration in the prediction of errors in radio navigational systems.

An excellent semiempirical approach to the problem of calculating fields over an inhomogeneous earth has been given by Millington [1949]. Sometime earlier Feinberg [1946] had formulated a general method for treating mixed paths over a flat earth. Similar results were obtained independently by Clemmow [1953] and Bremmer [1954]. The author [Wait, 1956a] showed that the integral formula for a two-section path could be easily derived by an application of the compensation theorem [Monteath, 1951]. It was also indicated in this paper [Wait, 1956a] that the results could be readily generalized to a spherical earth. In a further paper [Wait and Householder, 1957] extensive numerical results were given for propagation over a two-section path on a spherical earth. Furutsu [1956] has also considered propagation over a spherical earth under mixed-path conditions. His general method is based on solving the dual integral equations for the problem by an iterative procedure.

It is the purpose of this report to discuss the theory for mixed-path propagation over both two- and three-section paths on a smooth spherical earth. Some attention is given the various representations which may be used in practical field computations. The situation when one terminal is at great heights is considered in some detail since this solution describes the radiation pattern of an antenna when the ground plane is inhomogeneous. Where possible, a physical interpretation of the results is given.

2. Formulation for a Two-Section Path

The mutual impedance Z_{ab} between two vertical antennas located at A and B over a spherical earth of radius a is considered. The situation is illustrated in figure 1 where a

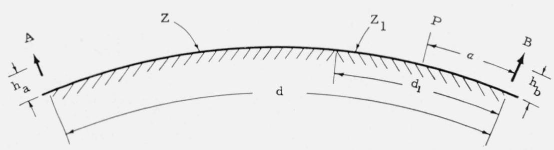

FIGURE 1. *Two-section path on a spherical earth.*

[1] Contribution from the Central Radio Propagation Laboratory, National Bureau of Standards, Boulder, Colo.

vertical cross section of the earth is shown. The great circle distance between A and B (measured along the surface of the earth) is d. The earth medium to the left of the boundary line has a conductivity σ and dielectric constant ϵ. The corresponding constants for the medium to the right of the boundary are σ_1 and ϵ_1. A variable P on the great circle path is a distance α from B. Then, for $\alpha > d_1$, the surface impedance is

$$Z \cong [i\mu_0\omega/(\sigma+i\epsilon\omega)]^{1/2}\left[1-\frac{i\epsilon_0\omega}{\sigma+i\epsilon\omega}\right]^{1/2} \tag{1}$$

while for $\alpha < d_1$

$$Z_1 \cong [i\mu_0\omega/(\sigma_1+i\epsilon_1\omega)]^{1/2}\left[1-\frac{i\epsilon_0\omega}{\sigma_1+i\epsilon_1\omega}\right]^{1/2} \tag{2}$$

where μ_0 is the permeability of the whole space which is assumed to be constant.

The mutual impedance between the dipoles A and B for the two-section mixed path illustrated in figure 1 is denoted Z'_{ab}. It was shown previously [Wait, 1956a] that it could be expressed in the form

$$I^2(Z'_{ab}-Z_{ab}) \cong (Z_1-Z)\iint_S (\vec{H}_{at}\cdot\vec{H}'_{bt})dS \tag{3}$$

where Z_{ab} is the mutual impedance if the surface of the earth were homogeneous with surface impedance Z *everywhere*. $\vec{H}_{at}$ is the tangential magnetic field of dipole A over the homogeneous earth while $\vec{H}_{bt}$ is the tangential magnetic field of dipole B over the inhomogeneous earth. The current in the dipoles are both taken equal to I for convenience. The surface of integration S extends over the region of the earth which is characterized by a surface impedance Z_1. Equation (3) also follows directly from the work of Monteath [1951].

3. Reduction to a One-Dimensional Problem

Equation (3) is essentially a two-dimensional integral equation for the fields. On a straightforward application of the principle of stationary phase, the surface integral can be reduced to a line integral from $\alpha=0$ to d_1 along the great circle path. Before stating this result it is convenient to introduce certain attenuation functions, W and W_1, as follows

$$Z_{ab}=\frac{l_a l_b i\mu_0\omega}{2\pi d}\,e^{-ikd}\,W(d,Z) \tag{4}$$

and

$$Z'_{ab}=\frac{l_a l_b i\mu_0\omega}{2\pi d}\,e^{-ikd}\,W'(d,Z,Z_1) \tag{5}$$

where l_a and l_b are the effective lengths of the dipoles A and B. It is to be understood that W and W' are also functions of h_a and h_b, the height of dipole A and B *above the ground*. The functions W and W' are normalized such that they would approach 1 if the earth were flat and perfectly conducting—and provided $h_a=h_b=0$. Consistently in what follows it is assumed that h_a and $h_b \ll d \ll a$.

The resulting one-dimensional integral equation is given by

$W'(d,Z,Z_1) \cong W(d,Z)$ for $d_1 < 0$

$$W'(d,Z,Z_1)\cong W(d,Z)-\left(\frac{ikd}{2\pi}\right)^{1/2}\left(\frac{Z_1-Z}{\eta_0}\right)\int_0^{d_1}\frac{W(d-\alpha,Z)W'(\alpha,Z_1,Z)}{[\alpha(d-\alpha)]^{1/2}}\,d\alpha \text{ for } d_1>0,$$

$$\text{where } \eta_0=(\mu_0/\epsilon_0)^{1/2}\cong 120\pi. \tag{6}$$

This result was obtained in an earlier paper [Wait and Householder, 1957] where certain numerical results were given for a two-section path in the frequency range 20 to 200 kc/s.

In the present work, it is desirable to introduce certain dimensionless parameters. These are defined by

$$iq=(ka/2)^{1/3}\,(Z/\eta_0),$$

$$iq_1=(ka/2)^{1/3}\,(Z_1/\eta_0),$$

$$x=(ka/2)^{1/3}\,(d/a),$$

$$\hat{x}=(ka/2)^{1/3}\,(\alpha/a),$$

$$x_1=(ka/2)^{1/3}\,(d_1/a).$$

Thus $W'(x,q,q_1)\simeq W(x,q)$ for $x_1<0$, and

$$W'(x,q,q_1)\simeq W(x,q)+\left(\frac{x}{\pi i}\right)^{1/2}(q_1-q)\int_0^{x_1}\frac{W(x-\hat{x},q)W'(\hat{x},q_1,q)}{[\hat{x}\,(x-\hat{x})]^{1/2}}\,d\hat{x} \tag{7}$$

for $x_1>0$. It is immediately observed that $W'(\hat{x},q_1,q)$ in the integrand may be replaced by $W(\hat{x},q_1)$ since $\hat{x}<x_1$ over the range of integration. Therefore

$$W'(x,q,q_1)\simeq W(x,q)+\left(\frac{x}{\pi i}\right)^{1/2}(q_1-q)\int_0^{x_1}\frac{W(x-\hat{x},q)W(\hat{x},q_1)}{[\hat{x}\,(x-\hat{x})]^{1/2}}\,d\hat{x}. \tag{8}$$

An alternate form of this equation is easily obtained by regarding the left-hand portion of the path as a modification to the homogeneous earth of surface impedance Z_1. Thus

$$W'(x,q_1,q)\simeq W(x,q_1)+\left(\frac{x}{\pi i}\right)^{1/2}(q-q_1)\int_0^{x-x_1}\frac{W(x-\hat{x},q_1)W(\hat{x},q)}{[\hat{x}\,(x-\hat{x})]^{1/2}}\,d\hat{x}. \tag{9}$$

4. Alternate Representations

Equations (8) and (9) are explicit integral formulas to permit the computation of the attenuation function W' in terms of the attenuation functions W appropriate for a homogeneous earth. The latter are well-known from the theory of van der Pol and Bremmer [1937] and Fock [1945]. From their work it follows that

$$W(x,q)=\left(\frac{\pi x}{i}\right)^{1/2}\sum_{s=1,2,3,\ldots}^{\infty}\frac{e^{-ixt_s}}{t_s-q^2}\,\frac{w_1(t_s-y_a)}{w_1(t_s)}\,\frac{w_1(t_s-y_b)}{w_1(t_s)} \tag{10}$$

where

$$x=(ka/2)^{1/3}\,(d/a),\qquad y_a=(2/ka)^{1/3}\,kh_a,\qquad y_b=(2/ka)^{1/3}\,kh_b.$$

The coefficients t_s are solutions of the equation

$$w_1'(t)-qw_1(t)=0 \tag{11}$$

where $w_1(t)$ is an Airy integral and the prime indicates a derivative with respect to t. In terms of Hankel functions of order one third, $w_1(t)=\exp\,(-2\pi i/3)(-\pi t/3)^{1/2}H^{(2)}_{1/3}(2/3)(-t)^{3/2}]$. The notation used here is closely akin to that used by Fock [1945].

In a similar fashion,

$$W(x,q_1)=\left(\frac{\pi x}{i}\right)^{1/2}\sum_{r=1}^{\infty}\frac{e^{-ixt_r^{(1)}}}{t_r^{(1)}-q_1^2}\,\frac{w_1(t_r^{(1)}-y_a)}{w_1(t_r^{(1)})}\,\frac{w_1(t_r^{(1)}-y_b)}{w_1(t_r^{(1)})} \tag{12}$$

where $t_r^{(1)}$ are solutions of

$$w_1'(t)-q_1w_1(t)=0. \tag{13}$$

Using (10) and (12), the integration in (8) may be readily carried out to yield

$$W'(x,q,q_1)-W(x,q)=\left(\frac{\pi x}{i}\right)^{1/2}(q_1-q)\sum_s\sum_r \frac{e^{-ixt_s}\left[e^{-ix_1(t_r^{(1)}-t_s)}-1\right]}{(t_r^{(1)}-t_s)(t_s-q^2)(t_r^{(1)}-q_1^2)}\frac{w_1(t_s-y_a)}{w_1(t_s)}\frac{w_1(t_r^{(1)}-y_b)}{w_1(t_r^{(1)})}. \tag{14}$$

The double summation converges quite rapidly if x_1 is of the order of unity or greater and provided that either y_a and y_b are not large compared with unity.

When x_1 is small, the double series expansion given in (14) becomes very poorly convergent. An alternate expansion for the case $h_b=0$ is obtained by using the following representation [Bremmer, 1953 and 1958; Wait, 1956].

$$W(\hat{x},q_1)=\sum_{m=0,1,2,\ldots} A_m e^{im\pi/4}q_1^m(\hat{x})^{m/2} \tag{15}$$

where

$$A_0=1,\quad A_1=-i\sqrt{\pi},\quad A_2=-2,\quad A_3=i\sqrt{\pi}\left(1+\frac{1}{4q_1^3}\right),\quad A_4=\frac{4}{3}\left(1+\frac{1}{2q_1^3}\right),$$

$$A_5=-\frac{i\sqrt{\pi}}{2}\left(1+\frac{3}{4q^3}\right),\quad A_6=-\frac{8}{15}\left(1+\frac{1}{q_1^3}+\frac{7}{32q_1^6}\right),\text{ etc.}$$

Using this result along with the residue series representation for $W(x-\hat{x},q)$ enables (8) to be expressed in the form

$$W'(x,q,q_1)=W(x,q)-ix^{1/2}(q_1-q)\sum_{s=1}^{\infty}\sum_{m=0}^{\infty}A_m e^{im\pi/4}q_1^m I_{\frac{m-1}{2}}\frac{w_1(t_s-y_a)}{w_1(t_s)} \tag{16}$$

where

$$I_{\frac{m-1}{2}}=\int_0^{x_1} e^{-i(x-\hat{x})t_s}(\hat{x})^{\frac{m-1}{2}}d\hat{x}. \tag{17}$$

The integral $I_{\frac{m-1}{2}}$ may be reduced by the following recurrence relation

$$I_{\frac{m}{2}}=\frac{1}{it_s}\left[e^{-i(x-\hat{x})t_s}(\hat{x})^{m/2}-\left(\frac{m}{2}\right)I_{\frac{m}{2}-1}\right]. \tag{18}$$

Successive application of this equation enables $I_{(m-1)/2}$ to be expressed in terms of I_0 and $I_{-1/2}$. These are given by

$$I_0=e^{-ixt_s}\int_0^{x_1}e^{i\hat{x}t_s}d\hat{x}=\frac{e^{-ixt_s}}{it_s}(e^{ix_1t_s}-1), \tag{19}$$

and

$$I_{-1/2}=e^{-ixt_s}\int_0^{x_1}\frac{e^{i\hat{x}t_s}}{(\hat{x})^{1/2}}d\hat{x}=\left(\frac{i\pi}{t_s}\right)^{1/2}e^{-ixt_s}\operatorname{erf}(\sqrt{-it_sx_1}) \tag{20}$$

where erf (Z) is the error integral of argument Z.

The double series expansion given by (16) is highly convergent if x is somewhat greater than 1 and x_1 is somewhat less than unity. In fact, if $|q_1^2x_1|\ll 1$, only the $m=0$ terms need be retained. Furthermore, if in addition $x_1\ll 1$,

$$I_{-1/2}\simeq 2e^{-ixt_s}(x_1)^{1/2}.$$

And thus,

$$W'(x,q,q_1)\simeq W(x,q)\left[1+\frac{q_1-q}{(\pi i)^{1/2}}2x_1^{1/2}\right]. \tag{21}$$

The term in square brackets can be regarded as the correction to the attenuation function as a result of the portion of the path from $\hat{x}=0$ to x_1. In terms of the original parameters of the problem, the square bracket term becomes

$$\left[1-\left(\frac{i}{\pi}\right)^{1/2}\frac{Z_1-Z}{\eta_0}(2kd_1)^{1/2}\right]. \tag{22}$$

This same correction emerged from the corresponding theory for the flat-earth case [Wait, 1956b].

5. Solution for One Antenna at Great Heights

When one terminal becomes elevated to a large height such that y_a or $y_b \gg 1$, it is desirable [following Fock, 1945] to replace the attenuation function W by an auxiliary function V. This function V actually characterizes the far zone radiation pattern of a vertical antenna on the curved surface [Wait and Conda, 1958]. First it is noted that if $|y-t|\gg 1$

$$w_1(t-y)\simeq e^{-i\pi/4}(y-t)^{-1/4}\exp\left[-i\frac{2}{3}(y-t)^{3/2}\right]. \tag{23}$$

Furthermore, if also $y\gg t$,

$$w_1(t-y)\simeq e^{-i\pi/4}y^{-1/4}e^{-i\frac{2}{3}y^{3/2}}e^{iy^{1/2}t}. \tag{24}$$

This latter asymptotic relation enables $W(x,q)$ to be written in the form

$$W(x,q)\simeq e^{-i\frac{2}{3}y_a^{3/2}}V(x-\sqrt{y_a},q)\frac{x^{1/2}}{y_a^{1/4}} \tag{25}$$

where

$$V(X,q)=-i\pi^{1/2}\sum_s\frac{e^{-iXt_s}}{(t_s-q^2)w_1(t_s)}\frac{w_1(t_s-y_b)}{w_1(t_s)}. \tag{26}$$

This suggests that the radiation pattern function $V'(X,q,q_1)$ for the mixed path be defined by the relation

$$W'(x,q,q_1)=e^{-i\frac{2}{3}y_a^{3/2}}V'(x-\sqrt{y_a},q,q_1)\frac{x^{1/2}}{y_a^{1/4}}. \tag{27}$$

The integral formula for V' may then be obtained directly from (8), thus

$$V'(X,q,q_1)=V(X,q)+\frac{(q_1-q)}{(\pi i)^{1/2}}\int_0^{x_1}\frac{V(X-\hat{X},q)W(\hat{X},q_1)}{(\hat{X})^{1/2}}d\hat{X} \tag{28}$$

where

$$X=x-\sqrt{y_a}\simeq\left(\frac{ka}{2}\right)^{1/3}\frac{d-\sqrt{2ah_a}}{a}.$$

It should be noted that X can be positive or negative depending on whether d is greater or less than $(2ah_a)^{1/2}$.

An alternate form for V' is obtained from (9), thus

$$V'(X,q,q_1)=V(X,q_1)+\frac{(q-q_1)}{(\pi i)^{1/2}}\int_{-\sqrt{y_a}}^{X_1}\frac{W(X-\hat{X},q_1)V(\hat{X},q)}{(X-\hat{X})^{1/2}}d\hat{X} \tag{29}$$

where $X_1=x_1-\sqrt{y_a}$. Since $y_a\gg 1$, the lower limit of this integral is effectively $-\infty$.

Series formulas for $V'(X,q,q_1)$ can be readily obtained in the manner discussed above for $W'(x,q,q_1)$. However, these are applicable only in the present situation if $X>0$ since the series given in (26) is divergent for $X<0$. A more suitable approach is to use the contour integral representation for $V(X,q)$. This is given by

$$V(X,q)=\frac{1}{2\pi^{1/2}}\oint\frac{e^{-iXt}}{w_1'(t)-qw_1(t)}\,\frac{w_1(t-y_b)}{w_1(t)}\,dt. \tag{30}$$

The contour is chosen so that it encloses (in a clockwise sense) the poles of the integrand at $t=t_s$. It is readily verified that $-2\pi i$ times the sum of the residues leads back to (26). It is now convenient to write

$$V(X,q)=\sum_{n=0}^{\infty}a_n(q)X^n \tag{31}$$

where

$$a_n(q)=\frac{e^{-in\pi/2}}{2\pi^{1/2}n!}\oint\frac{t^n}{[w_1'(t)-qw_1(t)]}\,\frac{w_1(t-y_b)}{w_1(t)}\,dt. \tag{32}$$

Methods for evaluating the coefficients $a_n(q)$ have been discussed by Logan [1959].

The above representation for V and the series form for the W given by (15) enable the integration in (28) for V' to be readily carried out. The result is

$$V'(X,\,q,q_1)-V(X,q)=\frac{q_1-q}{\pi i}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n(q)A_m(q_1)e^{im\pi/4}q_1^mP_{n,\frac{m-1}{2}} \tag{33}$$

where

$$P_{n,\frac{m-1}{2}}=\int_0^{d_1}(X-\hat{x})^n(\hat{x})^{\frac{m-1}{2}}\,d\hat{x}. \tag{34}$$

The integral $P_{n,\frac{m-1}{2}}$ can be reduced by using the following formula

$$P_{n,\frac{m}{2}}=\frac{(X-x_1)^n(x_1)^{\frac{m}{2}+1}}{\frac{m}{2}+1}+\frac{n}{\frac{m}{2}+1}P_{n-1,\frac{m}{2}+1}. \tag{35}$$

Successive applications enable $P_{n,\frac{m-1}{2}}$ to be expressed in terms of $P_{0,\frac{m}{2}}$ where

$$P_{0,\frac{m}{2}}=\int_0^{x_1}(\hat{x})^{m/2}d\hat{x}=\frac{(x_1)^{\frac{m}{2}+1}}{\frac{m}{2}+1}. \tag{36}$$

The above double-series representation for V' converges very rapidly when both $|X|$ and x_1 are reasonably small compared to 1.

6. Three-Section Path

The next most obvious extension is to a three-part medium. The situation is illustrated in figure 2. The approach used amounts to a successive application of the theory for the two-part medium. The path, between A and B, consists of three segments whose surface impedances are Z, Z_2 and Z_1. The length of the latter two segments are d_2 and d_1. Using the

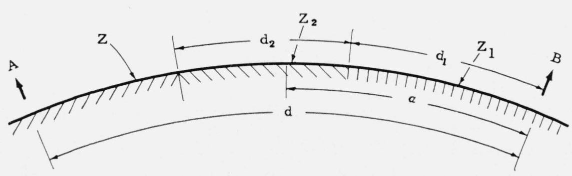

FIGURE 2. *Three-section path on a spherical earth.*

above-mentioned method, it really follows that the expression for the resultant attenuation function may be written

$$W'(d,Z,Z_2,Z_1)=W(d,Z)-\left(\frac{ikd}{2\pi}\right)^{1/2}\frac{Z_1-Z}{\eta_0}\int_0^{d_1}\frac{W(d-\alpha,Z)W(\alpha,Z_1)}{[\alpha(d-\alpha)]^{1/2}}\,d\alpha$$
$$-\left(\frac{ikd}{2\pi}\right)^{1/2}\frac{Z_2-Z}{\eta_0}\int_{d_1}^{d_1+d_2}\frac{W(d-\alpha,Z)W'(d,Z_1,Z_2)}{[\alpha(d-\alpha)]^{1/2}}\,d\alpha \qquad (37)$$

where $W(d,Z)$ is the attenuation function characteristic of propagation from A to B over a homogeneous earth of surface impedance Z. $W(d-\alpha,Z)$ and $W(\alpha,Z_1)$ are attenuation functions for propagation over homogeneous surfaces of surface impedances Z and Z_1, respectively. The integration variable α can be regarded as a great circle distance measured from B. The $W'(\alpha,Z_1,Z_2)$ occurring in the second integral is the appropriate attenuation function for propagation over the two-part medium from B to a point α on the middle segment.

For numerical work it is again convenient to introduce the dimensionless parameters used for the two-section theory. Thus

$$W'(x,q,q_2,q_1)=W(x,q)+\left(\frac{x}{\pi i}\right)^{1/2}(q_1-q)\int_0^{x_1}\frac{W(x-\hat{x},q)W(\hat{x},q_1)}{[\hat{x}(x-\hat{x})]^{1/2}}\,d\hat{x}$$
$$+\left(\frac{x}{\pi i}\right)^{1/2}(q_2-q)\int_{x_1}^{x_1+x_2}\frac{W(x-\hat{x},q)W'(\hat{x},q_1,q_2)}{[\hat{x}(x-\hat{x})]^{1/2}}\,d\hat{x} \qquad (38)$$

where

$$iq_2=(ka/2)^{1/3}(Z_2/\eta_0), \qquad x_2=(ka/2)^{1/3}(d_2/a),$$

and the other quantities have their usual meaning. From (8) and (9) we may deduce that

$$W'(\hat{x},q_1,q_2)=W(\hat{x},q_1)+\left(\frac{x}{\pi i}\right)^{1/2}(q_2-q_1)\int_0^{\hat{x}-x_1}\frac{W(\hat{x}-x',Z_1)W(x',Z_2)}{[(\hat{x}-x')x']^{1/2}} \qquad (39)$$

$$=W(\hat{x},q_2)+\left(\frac{x}{\pi i}\right)^{1/2}(q_1-q_2)\int_0^{x_1}\frac{W(\hat{x}-x',Z_2)W(x',Z_1)}{[(\hat{x}-x')x']^{1/2}}\,dx'. \qquad (40)$$

Of course it would be possible to obtain various series formulas for W' for a three-section path, but in general these would be cumbersome. It is considered to be preferable to work directly with (38) and carry out the integration numerically since presumably $W'(\hat{x},q_1,q_2)$ would be known from the two-section theory.

There is a rather simple limiting case of (38) which is rather interesting. We take $q=q_1$ and $x_2 \ll x$ and x_1. To this approximation, the first integral on the right-hand side of (38) vanishes and the integrand in the second integral is essentially a constant over the range x_1 to x_1+x_2. Thus

$$W'(x,q,q_2,q_1)\cong W(x,q)+\left(\frac{x}{\pi i}\right)^{1/2}(q_2-q)\frac{x_2}{[x_1(x-x_1)]^{1/2}}W(x-x_1,q)W(x_1,q). \qquad (41)$$

This simple formula has a clear physical interpretation. The second term on the right-hand side can be regarded as the field scattered by the strip of width x_2. It is proportional to q_2-q (or Z_2-Z) which is the surface impedance contrast and it also is proportional to the attenuation functions $W(x-x_1,q)$ and $W(x_1,q)$ which respectively account for the attenuation from A to the strip, and from the strip to B. Such an equation is analogous to the first Born approximation in scattering theory.

In the foregoing discussion of the three-section theory the antenna heights h_a and h_b, of A and B respectively, may be arbitrary (provided that h_a and $h_b \ll a$). When the height of one terminal is large, such that $y_a \gg 1$, for example, it is again convenient to define a radiation pattern function V'. In this case,

$$W'(x,q,q_2,q_1)=e^{-i\frac{2}{3}y_a^{3/2}}V'(X,q,q_2,q_1)\frac{x^{1/2}}{y_a^{1/4}} \tag{42}$$

where

$$V'(X,q,q_2,q_1)=V(X,q)+\frac{(q_1-q)}{(\pi i)^{1/2}}\int_0^{x_1}\frac{V(X-\hat{x},q)W(\hat{x},q_1)}{(\hat{x})^{1/2}}d\hat{x}$$

$$+\frac{(q_2-q)}{(\pi i)^{1/2}}\int_{x_1}^{x_1+x_2}\frac{V(X-\hat{x},q)W'(\hat{x},q_1,q_2)}{(\hat{x})^{1/2}}d\hat{x} \tag{43}$$

where all quantities on the right-hand side have their usual meaning.

7. Some Extensions of the Theory

It is of interest to apply the compensation theorem as expressed by (3) to a spherical earth in the case when the surface impedances are a function of distance along the great circle distance d between A and B. The one-dimensional form of (3) may be written in terms of the attenuation functions as follows

$$W'(q,q_1,x)=W(q,x)+\left(\frac{x}{\pi i}\right)^{1/2}\int_0^{x_1}[q_1(\hat{x})-q(\hat{x})]\frac{W(x-\hat{x},q)W'(\hat{x},q_1,q)}{[\hat{x}(x-\hat{x})]^{1/2}}d\hat{x} \tag{44}$$

where q_1 and q are now functions of $\hat{x}$ (or α) the great circle distance from B. Of course, if q_1 and q are constant, (44) is identical to (7).

It is believed that (44) could be solved directly by numerical means if q is taken as a constant corresponding to the unmodified homogeneous earth and $q_1(\hat{x})$ is some specified function of $\hat{x}$ over the range from 0 to x_1. Equation (44) also permits an approximate treatment of transition regions. To illustrate this latter point we consider a three-section path as shown in figure 2. Now, however, the middle section is considered to have a surface impedance which is a function of α. Thus, q_2 is a function of $\hat{x}$ over the region from x_1 to x_1+x_2. The attenuation function for this particular path may be expressed by

$$W'(x,q,q_2(x),q_1)=W'(x,q,q_1)+\left(\frac{x}{\pi i}\right)^{1/2}\int_{x_1}^{x_1+x_2}[q_2(\hat{x})-q]\frac{W(x-\hat{x},q)W'(\hat{x},q_1,q_2(x),q)}{[\hat{x}(x-\hat{x})]^{1/2}}d\hat{x} \tag{45}$$

where $W'(x,q,q_1)$ is the appropriate two-section attenuation function and is given explicitly by (8). Now on the assumption that x_2 is small compared with both x_1 and x_2 it is permissible to replace W' in the integral in (45) by $W(x_1,q_1)$. Admittedly, this is a first-order approximation but it should be adequate since the relative contribution of the integral is expected to be small in any case. To the same approximation $W(x-\hat{x},q)$ in the integrand can be replaced by $W(x-x_1,q)$. Thus

$$W'(x_1,q,q_2(x),q_1)\cong W'(x,q,q_1)+\left(\frac{x}{\pi i}\right)^{1/2}(\overline{q}_2-q)\frac{x_2}{[x_1(x-x_1)]^{1/2}}W(x-x_1,q)W(x_1,q_1) \tag{46}$$

where

$$\overline{q}_2=\frac{1}{x_2}\int_{x_1}^{x_1+x_2}q_2(\hat{x})dx \text{ is the average value of } q_2(\hat{x}).$$

8. Appendix

In the main text of the paper the attenuation function W for a homogeneous earth consistently appears. It is of interest to see that an integral equation for W in the case $h_a=h_b=0$ emerges directly from (44). To show this we allow x_1 to approach zero in (44) and then we imagine the unmodified surface to be a flat, perfectly conducting plane between A and B.

Thus, the appropriate value $W(x,q)$ is simply exp $(ix^3/12)$. (The quantity $x^3/12$ is the difference between the arc and the chord between A and B.) Furthermore, the appropriate value of the surface impedance along the arc AB *over* a conducting plane is easily found to be given by

$$Z \cong -\eta_0 \frac{d-\alpha}{2a}$$

or

$$q(\hat{x}) \cong i(x-\hat{x})/2.$$

Thus

$$W'(x,q_1(x)) = e^{ix^3/12} + \left(\frac{x}{\pi i}\right)^{1/2} \int_0^x \left[q_1(x) - i\,\frac{(x-\hat{x})}{2}\right] e^{i(x-\hat{x})^3/12} W'(\hat{x},q_1(\hat{x}))d\hat{x}. \tag{47}$$

Now if q_1 is a constant throughout the length of the path, W' in the above equation becomes by definition the attenuation function for a homogeneous path. Thus

$$W(x,q_1) = e^{ix^3/12} + \left(\frac{x}{\pi i}\right)^{1/2} \int_0^x \left[q_1 - i\,\frac{(x-\hat{x})}{2}\right] [\hat{x}(x-\hat{x})]^{-1/2} e^{i(x-\hat{x})^3/12} W(\hat{x},q_1)d\hat{x}. \tag{48}$$

This integral equation has been derived by Hufford [1952] who also shows that its solution is the residue series

$$W(x,q_1) = \left(\frac{\pi x}{i}\right)^{1/2} \sum_{r=1,2,3\ldots} \frac{e^{-ixt_r}}{t_r - q_1^2} \tag{49}$$

where t_r are roots of

$$w_1'(t) - q_1 w_1(t) = 0.$$

In a similar manner it may be shown that

$$V(X,q_1) = e^{iX^3/3} + \left(\frac{i}{\pi i}\right)^{1/2} \int_{-\infty}^X \left[q_1 - i\,\frac{(X-\hat{X})}{2}\right] [X-\hat{X}]^{-1/2} e^{i(X-\hat{X})^3/12} V(\hat{X},q_1)d\hat{X} \tag{50}$$

where

$$V(X,q_1) = -i\pi^{1/2} \sum_{r=1,2,3\ldots} \frac{e^{-iXt_r}}{(t_r - q_1^2)w_1(t_r)}. \tag{51}$$

These particular integral equations have also been discussed by Logan [1959].

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Wait, J. R., and J. Householder, Mixed path ground wave propagation, 2. Larger distances, J. Research NBS **59,** 19–26 (July 1957) RP2770.

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Publications of the Staff of National Bureau of Standards*

Selected Abstracts

Two magneto-ionic phenomena permitting the observation of valley minima between the E and F regions in the Arctic, J. W. Wright, *pp. 85–93 of* **Some ionospheric results obtained during the International Geophysical Year;** *Proc. symp. organized by the URSI/AGI committee, Brussels, 1959, edited by W. J. G. Beynon (Elsevier Publ. Co., Amsterdam, 1960).*
Two phenomena are described which result from the influence of electron collisions on radio wave reflections from the E region at high magnetic latitudes. One permits observation of the depth of the "valley" between the E and F regions; the other permits the measurement of electron density at a fixed and known height in this valley, when it exists.

The CRPL electron density profile programme: Some features and early results, J. W. Wright, *pp. 215–220 of* **Some ionospheric results obtained during the International Geophysical Year;** *Proc. symp. organized by the URSI/AGI committee, Brussels, 1959, edited by W. J. G. Beynon (Elsevier Publ. Co., Amsterdam, 1960).*
Electron density data and other quantities ($h_{\max}F2$, scale height, total electron content) provided by reduction of vertical soundings to N(h) profiles at CRPL, are described. A model of the F region above $h_{\max}F2$ is developed, corresponding to a Chapman profile of 100 km scale height, and is shown to be in agreement with available rocket and moon-echo data. Vertical cross sections of the ionosphere across the geomagnetic equator are illustrated and discussed.

Pecularities of the ionosphere in the Far East: Sporadic *E* and *F* region scatter, E. K. Smith, Jr., and J. W. Finney, *pp. 182–191 of* **Some ionospheric results obtained during the International Geophysical Year;** *Proc. symp. organized by the URSI/AGI committee, Brussels, 1959, edited by W. J. G. Beynon (Elsevier Publ. Co., Amsterdam, 1960).*
Through a study of ionosonde data from the world-wide network of stations and of miscellaneous oblique-incidence field-strength measurements made in Japan and the United States it was discovered shortly before the beginning of the IGY that sporadic E appears considerably more intense in the Far East than in similar latitudes in the Western Hemisphere. Comparisons of sporadic E data are very difficult unless made under identical conditions. Therefore an experiment was designed which consisted of recording transmission loss over two matched 50 Mc/s oblique-incidence circuits of approximately 800 miles in length, one in the Far East and the other in the Caribbean. The principal advantage of an oblique-incidence circuit over a vertical-incidence one at the equivalent frequency (about 11 Mc/s in this case) is that although the amount of sporadic E observed would be comparable, the effect of D region absorption in decibels at 50 Mc/s would be less than at 11 Mc/s by a factor of about 4.5.

Studies of scattering phenomena in the equatorial ionosphere based upon VHF transmissions across the magnetic equator, K. L. Bowles and R. Cohen, *pp. 192–195 of* **Some ionospheric results obtained during the International Geophysical Year;** *Proc. symp. organised by the URSI/AGI committee, Brussels, 1959, edited by W. J. G. Beynon (Elsevier Publ. Co., Amsterdam, 1960).*
Results have been attained pertaining to the phenomena of equatorial sporadic E, equatorial spread F, and propagation by means of ionospheric radio wave scattering at VHF near the magnetic equator. These studies are the results of IGY experimentation in South America utilizing 50 Mc/s transmissions crossing the magnetic equator at Huancayo, Peru, and over other oblique paths. Additional ionospheric experiments were also conducted in Huancayo at vertical incidence. It has been established that a close relationship exists between the magnetic manifestations of the equatorial electrojet current above Huancayo, the occurrence on Huancayo ionograms of equatorial sporadic E, and the intensity and fading rate of VHF signals propagated by the ionosphere over Huancayo. The zone in which E region effects of the equatorial current-stream are apparent extends over some 10° of latitude. Also, the scatter propagation in the equatorial ionosphere is characterized by higher signal-levels and by more rapid fading-rates than are observed in the ionosphere at temperate latitudes, and signal-strengths remain high throughout the night. The feasibility of utilizing ionospheric scattering for communications in equatorial regions seems quite promising. Curiously, ionospheric scatter propagation at VHF similar to that associated with sporadic E formations in temperate latitudes is observed from time to time just to the North and South of the magnetic equator, but its occurrence appears to be excluded in the immediate vicinity of that equator. Further a remarkable sensitivity to polarization was established in HF radar echoes obtained at vertical incidence from the diffuse region (bounded by the slant sporadic E and extending to the q-type sporadic E), comprising the typical magnetic-equatorial sporadic E configuration appearing on Huancayo ionograms. These diffuse echoes were obtained with an antenna-array polarized magnetic North-South, but were virtually eliminated upon using an East-West polarized antenna-array for transmitting and/or receiving.
Ionospheric propagation by scattering from the F region was sought over a transequatorial path employing a 2580 km transmitter-receiver separation. Propagation by F-scatter was present only about 10 percent of the time, and this was during evening hours, when enhancements in signal-strength above the levels due to E region propagation were found to occur over this circuit, generally before midnight. A sufficient and nearly necessary condition for the occurrence of such openings was found to be the presence of equatorial spread F indications on the Huancayo ionograms. By the variation of antenna radiation patterns and by oblique incidence pulse-delay measurements, the height of the propagation medium sustaining this F-scatter was established, in general, to be identifiable with the *lowest* height of the associated equatorial spread F seen on Huancayo ionograms. Thus, the spread-F time delays are not the result of an intervening screen below the F region that is invisible on the ionograms, as had been suggested in certain theoretical explanations. Further, the received pulses over the oblique path were but little wider than those transmitted. This lack of pulse-broadening and the narrowing of the equatorial spread F configuration on the Huancayo ionogram upon using a high-gain antenna-array there at vertical incidence, indicate that the equatorial spread F consists of a thin sheet of irregularities. From the pulse measurements, the thickness of this sheet is estimated at the order of 50 km. The not always perfect comparisons in the lowest heights of Huancayo spread F with heights deduced from pulse-delay in the F-scatter is interpreted as evidence for a limiting geographical extension of a given scattering sheet in the F region. Polarization measurements similar to those described above prove indirectly that the equatorial spread F echoes observed at Huancayo must arrive from the magnetic East-West direction; i.e., that orthogonality to the magnetic field is imposed as a requirement for obtaining them. This is good evidence that the irregularities are elongated in the North-South direction, along the earth's magnetic lines of force. Thus, the time delays associated with equatorial spread F at Huancayo must arise from echoes to the East and West of the zenith. From the success in obtaining scattering at 50 Mc/s up to heights of 500 km or more, it may be inferred that irregularities of scale size 10 metres or smaller, measured normal to the magnetic field lines, can exist up to these heights at night in the equatorial F region.

Standard frequencies and time signals from NBS stations WWV and WWVH, *by* Radio Standards Laboratory, *NBS Misc. Pub. 236* (*Dec. 1, 1960*).
Detailed descriptions are given of six technical services broadcast by National Bureau of Standards radio stations WWV and WWVH. The services include 1, standard radio frequencies; 2, standard audio frequencies; 3, standard time intervals; 4, standard musical pitch; 5, time signals; and 6, radio propagation forecasts. Other domestic and foreign standard frequency and time signal broadcasts are tabulated.

Near infrared atmospheric transmission to solar radiation, D. M. Gates, *J. Opt. Soc. Am.* **50,** *No. 12, 1299–1304* (*Dec. 1960*).
Near infrared solar spectrum observations taken on October 15, 1954 with a double-pass NaCl prism spectrometer have been analyzed for transmission coefficients for the "selective" absorption factor and for the "continuum" factor. The analysis was carried out for 59 wavelength positions between 0.872 and 2.537μ. The monochromatic data fit well the law $lnT = c_1(w)^{1/2}$ where w is the amount of water vapor in the optical path. The coefficient c_1 is given as a continuous function of the wavelength. A coefficient of extinction for the "continuum" factor is also given.

Limitations of radiosonde punch-card records for radio-meteorological studies, B. R. Bean and B. A. Cahoon, *J. Geophys. Research* **66,** *No. 1, 328–331* (*Jan. 1961*).
Instrumental and data reduction restrictions affect the conversion of radiosonde data into radio refractive index profiles. It is found that over half of the elevated radio ducts reported by the observer will be missed due solely to the standard criteria established for rejection of data points in the process of preparing punch-card records.
An extensive collection of radiosonde data containing all observed levels is now available from some seventy locations on and near the North American Continent.

VLF phase perturbation associated with meteor shower ionization, C. J. Chilton, *J. Geophys. Research* **66,** *No. 2, 379–383* (*Feb. 1961*).
Anomalies associated with the Lyrid, δ-Aquarid, and Perseid meteor showers were observed on the daily phase variation of the 16 kc/s transmission from Rugby, England, to Boulder, Colorado. The effective reflection heights estimated from these anomalies are 80, 81, and 83 km respectively.

Microwave spectroscopy—Atomic frequency standards, J. M. Richardson, **Encyclopedia of Spectroscopy,** *edited by G. L. Clark* (*Reinhold Publ. Corp., New York, N.Y., 1960*).
The topic of atomic frequency standards is surveyed in a general way with emphasis on principles and current results. Atomic transitions satisfy very well certain requirements of a good standard; namely, constancy, precision, renewability, and convenience. Technique of observation, results, and limitations of several types of atomic frequency standards are described. These are the gas absorption, atomic beam, maser, and optical pumping methods. The atomic resonance may be used as a resonator or as a frequency controlling element in a control mechanism. Useful applications are improvement in the time scale, analysis of power spectra and line shapes, and tests of certain postulates of special and general relativity.

A theoretical study of sporadic-*E* structure in the light of radio measurements, K. Tao, *NBS TN87* (*PB161588*) (*$1.25*).
The theoretical aspects of the mechanisms of sporadic-*E* reflections are described from both the standpoint of a thin layer and a scattering model. For the thin layer model, thin dielectric layers which have various distributions of electron density are considered. It is also pointed out that the scattering theory for which an autocorrelation function of the fluctuation of electron density is given by modified Bessel functions of the fourth through the seventh order is an available model for sporadic-*E* scatter. Moreover blobs of ionization which have a horizontal scale of the order of 200 m and a vertical scale of about 50 m are considered for sporadic-*E* scatter. The frequency and distance dependences of the oblique VHF propagation by means of the sporadic-*E* layer are discussed by comparing the theoretical results with experimental evidence.

The NBS meteor-burst propagation project—a progress report, C. E. Hornback, L. D. Breyfogle, and G. R. Sugar, *NBS TN86* (*PB161587*) (*1960*) *$1.25*.
This report briefly describes a meteor-burst propagation-study program at NBS-Boulder Laboratories and presents some of the preliminary analysis results. Observations have been made with scaled systems over three different paths (Long Branch-Table Mesa, Norman-Fargo, and Barrow-Kenai) at frequencies of 30, 50, and 74 Mc/s. The recorded data is processed by a combination of manual and automatic methods. The preliminary results show about a 10-db diurnal variation in threshold for a constant duty-cycle. Thresholds for a constant duty-cycle were observed to have an approximate frequency dependence relative to 30 Mc/s of 15 db lower for 50 Mc/s and 30 db lower for 74 Mc/s. There was no statistically-significant difference observed in the occurrence of meteor-bursts from a Poisson distribution.

Oblique incidence receiving antenna array for a relative ionospheric opacity meter, A. C. Wilson, *NBS TN78* (*PB161579*) (*Nov. 1960*) *50 cents*.
Experimental measurements incidental to the design of an antenna for a relative ionospheric opacity meter (RIO Meter) are described.
The frequency of operation is 50 Mc. The antenna requirements are that the main lobe of the antenna is directed at 23° above the horizon, the half-power beam-width in the vertical plane does not exceed 10°, the minimum front-to-back ratio is at least 13 decibels, and the side-lobe levels are at least 10 decibels below the maximum response in both the E- and H-planes. Since the antenna is for use in Alaska, it is to be of simple design and physically able to withstand any anticipated wind and ice loads.
The final antenna design is an array of three stacked horizontal dipoles with two optimally spaced reflectors behind each dipole to obtain the required directivity. The narrow main lobe of the antenna directed at an angle of 23° above the horizon is obtained by properly spaced and phased dipoles above the ground. The half-power beamwidth in the vertical plane is computed to be 7¼°. The front-to-back ratio over the rear 180° sector is not less than 20 decibels, and the half-power beamwidth in the E-plane is 74°.
Two complete receiving antenna arrays were constructed, adjusted, and tested. These antennas were installed in Alaska where they are now in use by an auroral transmission loss project.

A study of auroral coruscations, W. H. Campbell and M. H. Rees, *J. Geophys, Research* **66,** *No. 1, 41–55* (*Jan. 1961*).
Short period variations in the $N_2+(0, 0)$ auroral emission band represent 5 per cent of the total light in the 3914A region. These quasi-periodic coruscations have a dominant period of 6 to 10 seconds. They attain a maximum amplitude in the predawn hours and are closely related to magnetic field micropulsations and ionospheric absorption of cosmic noise. Spectroscopic triangulation showed that the variations originate in the *E* region of the ionosphere. The profile of the electron density associated with the aurora was found to have a maximum value of 1.1×10^6 electrons per cubic centimeter at 98 kilometers.

Reports on CSAGI Disciplines, Part Ai, The Airglow, *by* F. E. Roach. **Annals of the International Geophysical Year, X** 134–137 (*Pergamon Press, New York, 1960*).
Hourly zenith intensities of the night airglow in rayleighs for IGY and IGC. Results from 28 stations.

Propagation of electromagnetic pulses in a homogeneous conducting earth, J. R. Wait. *Appl.Sci. Research, Section B,* **8,** *213–253* (*1960*).
A general analysis for the electromagnetic response of conducting media due to pulse excitation is presented. The treatment is based on the Laplace transform theory. First, a survey of the field is made and the limitations and scope of the previous work are pointed out. The theory of propagation of a plane wave pulse in a conducting and homogeneous medium of infinite extent is then reviewed. The form of these results enable one to evaluate the relative importance of the conductivity and the dielectric constant. It is indicated, for sufficiently large times in the transient response, that displace-

ment currents may be safely neglected for sea water and for most geological media. Under this assumption, the waveform of the electric field in a conducting medium is illustrated for the case where the source is an electric dipole energized by a step-function current. Results are also presented for exponential and bell-shaped source functions. The pulse shape of the field components is profoundly modified as they propagate through the medium. It is suggested that this property may be utilized in measuring distances in the earth's crust. The more difficult problem of propagation in non-infinite conducting media is also considered. To account for the presence of the interface in a conducting half space (i.e., homogeneous flat ground), a rather involved analytical expression for the transient fields is required. Certain special cases, such as a horizontal electric dipole at the interface, are illustrated by numerical results. The transient excitation of a wire loop lying on the surface of a homogeneous ground is also considered. Finally, transient coupling between pairs of parallel insulated wires grounded at their end points is treated as an extension of the earlier results.

Ionospheric absorption at times of auroral and magnetic pulsations, W. H. Campbell and H. Leinbach, *J. Geophys. Research* **66,** *No. 1, 25–35 (Jan. 1961).*
A study in March and April 1960, showed variations in the auroral zone ionospheric absorption of cosmic noise to be closely related to magnetic field micropulsations and short period coruscations of λ 3914. At times of polar-cap type absorption, magnetic micropulsation amplitudes were diminished. Auroral ionization in the E region, estimated from a particular luminosity-height profile, accounted for 50 per cent, at least, of the cosmic noise absorption.

Tests for regression coefficients when errors are correlated, M. M. Siddiqui. *Annals of Math. Stat.* **31,** *No. 4, 929–938 (Dec. 1960).*
In a previous paper the covariances of least-squares estimates of regression coefficients and the expected value of the estimate of residual variance were investigated when the errors are assumed to be correlated. In this paper we will investigate the distribution of the usual test statistics for regression coefficients under the same assumptions. Applications of the theory to the cases of testing a single sample mean, the difference between the means of two samples, the coefficients in a linear trend and in regression on trigonometric functions will be discussed in some detail under an assumed covariance matrix for errors.

Climatic charts and data of the radio refractive index for the United States and the World, B. R. Bean, J. D. Horn, and A. M. Ozanich, Jr., *NBS Mono. 22 (Nov. 25, 1960) $2.00.*
The radio refractive index of air, $n=1+N\times10^{-6}$, is a function of atmospheric pressure, temperature, and humidity and varies in a systematic fashion with climate.
Included in this Monograph is a compilation of refractive index data. Data listings made up of observations from 45 U.S. surface weather stations for 2-hour intervals over an 8-year period are given. Mean values, maxima, minima, and standard deviations of the refractive index have been calculated and tabulated for these observations. Additionally, mean vertical profiles of the refractive index have been prepared for 43 U.S. upper air sounding stations from long-term means of pressure, temperature, and humidity.
Earlier studies of refractive index climate are assimilated and put into perspective. One such study is an extensive analysis and mapping of the refractive index climate of the United States. A worldwide radio refractive index climatology is developed based upon monthly mean observations of pressure, temperature, and humidity.
An important finding of these climatological investigations is the strong correlation of N with height. A reduced-to-sea-level value of the index, termed N_0, is used to eliminate this systematic height dependence. The surface value of N, N_s, may be estimated four to five times more accurately from charts of N_0 than from similar-sized charts of N_s itself.
From climatic charts of N_0, N_s may be estimated at any given location in the United States throughout the day during every season. In addition detailed annual and diurnal cycles, as well as 8-year cumulative probability distributions, are given for 12 representative U.S. stations.
On a worldwide basis, charts of mean N_0 are presented for both summer and winter season.

Diffractive corrections to the geometrical optics of low-frequency propagation, J. R. Wait. **Electromagnetic Wave Propagation (International Conference Sponsored by the Postal and Telecommunications Group of the Brussels Universal Exhibition)**, *edited by M. Desirant and J. L. Michiels, pp. 87–101 (Academic Press, New York, N.Y., 1960).*
The influence of caustics in low frequency radio propagation is studied. Correction factors to be applied to the geometrical optical representations are presented in the form of graphs. Particular attention is paid to the treatment of the caustic which occurs at the geometrical horizon of the first hop sky wave. It is shown that the apparent infinity in the convergence coefficient can be removed when the wave nature of the problem is considered. Antipodal phenomena are also discussed.

NOTE. In this paper the publisher deleted the captions for Figures 7a, 7b, 8a, and 8b. These should read:

Fig. 7a—The factor $|F|$ for sea water, $\epsilon=80\epsilon_o$, $\sigma=5$ mhos/meter, $a=4/3\times4360$ km.

Fig. 7b—The phase, of the factor F, for sea water.

Fig. 8a—The factor $|F|$ for average land, $\epsilon=15\epsilon_o$, $\sigma=5\times10^{-3}$ mhos/meter, $a=4/3\times4360$ km.

Fig. 8b—The phase Δ, of the factor F, for average land.

Furthermore, in these four figures the abscissae should be labelled "Angle below horizon $(\theta-\theta_c)/2$." Also, in equation (25) the factor (F) is defined by

$$(F)=\frac{|F|}{2}\,e^{i\Delta}$$

where $|F|$ and Δ are the quantities plotted in Figures 7 and 8.

Propagation of electromagnetic waves along a thin plasma sheet, J. R. Wait, *Can. J. Phys.* **38,** *1586–1594 (Dec. 1960).*
It is shown that a thin ionized sheet will support a trapped surface wave. The effect of a constant and uniform magnetic field is to modify the phase velocity and polarization of the surface wave. The essential features are illustrated by numerical results for selected values of the electron density, collision frequency, and gyro frequency. The effect of locating the plasma sheet near and parallel to a conducting plane is also considered. In this situation other modes of a waveguide type are possible in addition to the surface wave.

Ionospheric mapping by numerical methods, W. B. Jones and R. M. Gallet, *Telecomm. J., Journal UIT, No. 12, 260e–264e (Dec. 1960).*
One of the most difficult problems encountered in forecasting conditions for long-distance radio propagation is the mapping of ionospheric characteristics. A satisfactory method has been developed for mapping the continuous time variations on a world-wide basis of any ionospheric quantity entirely by numerical methods and electronic computing.
As an illustration, the method is applied to mapping the monthly median values of the critical frequency of the F2 layer (foF2) for December 1957. Input to the computer consists of the data as they are tabulated at 109 available stations. An analysis is performed by progressively fitting series of orthogonal functions in three coordinates: time, latitude, and longitude. Two main difficulties have been overcome: (1) the data were irregularly positioned in the two space dimensions, and (2) they were affected by statistical noise. The end product of the analysis is a numerical map, or mathematical function, defined by a table of coefficients, small in comparison with the number of input data.
Many applications can be made using the table of coefficients. For example, a variety of graphical representations and statistical tests for accuracy may be produced, as well as forecasting ionospheric conditions using a suitable solar index. The numerical methods are general enough to be applied to any geophysical quantity much as meteorological or geomagnetic characteristics varying continuously with time on a world-wide basis.

Other NBS Publications

Journal of Research, Vol. 65A, No. 3, May-June 1961. 70 cents.

International practical temperature scale of 1948. Text revision of 1960, H. F. Stimson.
Evaluation of the NBS unit of resistance based on a computable capacitor, R. D. Cutkosky.
Wavelengths and intensities in the first spectrum of bromine, 2000 to 13000 A, J. L. Tech and C. H. Corliss.
Torsional resonance vibrations of uniform bars of square cross section, W. E. Tefft and S. Spinner.
Infrared studies of aragonite, calcite, and vaterite type structures in the borates, carbonates, and nitrates, C. E. Weir and E. R. Lippincott.
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Pyrolysis of fluorocarbon polymers, L. A. Wall and S. Straus.
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